Basic concepts for a theory of evaluation: The aggregative operator

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Starting with an explication of the "aggregative"-concept and deducing a general structure which satisfies a number of minimal requirements (properties of clustering) the main features of a new mathematical theory – called "theory of evaluation" – are developed. The theory sheds new light on such well-known concepts as membership, conjunction and disjunction and seems to be a very promising tool to handle representation problems as they grow from the fields of theory of fuzzy set, and its many applications, of human decision making and of multicriteria analysis.

Introduction

The importance of the study of fuzzy sets perhaps presented itself first in the theory of multicriteria decisions. Even the first publication (L.A. Zadeh, 1965) includes a possible model of a decision problem. The fuzzy set can be regarded as a utility function. Further, fuzzy sets may prove adequate means for a mathematical description of the concept of descriptive decisions. Such models would enable the gulf between descriptive and normative decisions to be eliminated.

The theory of fuzzy sets has come into full blossom in the past few years, but the fruition awaited by so many researchers has not yet been attained. The reason for this is well illustrated by some of the criticisms that the theory of the fuzzy sets has come in for. I should like to mention a few critical comments without giving a detailed account of the theory.

1. Some critical aspects of the theory of fuzzy sets

1.1. The problem of interpreting the membership function

Zadeh's generalization of sets, which utilizes the isomorphism of the set and its characteristic function, has proved very simple and clear. However it is here that the first problem arises: How should the membership function be interpreted? Zadeh's aim was primarily the mathematical modelling of everyday concepts that are not sharply defined. It is precisely this practice-oriented interpretation that is problematic, as the concepts used in communication have not facilitated a homogeneous interpretation. Let us examine these considerations on concrete examples, such as the ideas "a red apple", "the bus runs often", and "a young factory".

(a) In the first example, the word "red" in terms of the apple can be interpreted as the ratio of the red surface area to the total surface area of the apple and it has nothing to do with the fuzzy set "red".

(b) In the second case, the word "often" suggests a probability, or rather a subjective probability interpretation.

(c) The third example is perhaps the most typical. To date attention has been paid merely to the interpretation of the word "young" itself, and it has tacitly been assumed that the word refers to humans. It may be seen from this example that a membership function assigned to such fuzzy concepts is different when it relates to different objects. Besides the interpretation involving only time, the expression "a young factory" can also imply that it has not yet acquired sufficient experience, or the possibility that its expansion and development is still incomplete.

Rather than extending this list of examples, we may state that it is not a task of the theory of fuzzy sets to provide a comprehensive description of linguistic powers of expression; nevertheless, it is absolutely essential to grasp that aspect of the language which it is desired to model.

1.2. The problem of the operators

As we have already mentioned, the generalization proposed by Zadeh is based on isomorphous operations that can be interpreted for a set and for its characteristic function. Zadeh identified mini-

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mum and maximum operators interpreted for the characteristic function with the operation of set intersection and set union. Empirical investigations have not provided much support for the usefulness of these operators. It is clearly not advisable to make the construction of a theory dependent on the empirical results. The mathematical theories solve this problem by means of the axiomatic method, which can then be compared with empirical results. It is naturally possible to give an axiomatic system whose solution is the above-mentioned operators, but essentially this does not alter the situation.

As regards the various operators corresponding to other set theory operations, I should like to mention the entailment relation of the sets. The Zadeh formulation is as follows:

$$A \subseteq B \text{ if } \mu_A(x) \leq \mu_B(x) \quad (x \in X) \qquad (*)$$

The definition (*) appears fairly natural, but whereas the result of the operation of set intersection and union is likewise a fuzzy set, which may be said to be evaluated in many-valued logic, (*) is evaluated in dual logic. In addition, the implication may be derived with the aid of the introduced complementary and intersection (or union) operations. Overall, therefore, set implication does not form a uniform system with the other set theory operations introduced. This should be noted, for even if the implication does not play such a significant role in the theory of fuzzy sets, the consequence is all the more a concept in fuzzy logic.

1.3. The problem of the fuzzy decision

It was stated earlier that the aim of the theory of fuzzy sets was the description of human systems. The most fundamental problem in this field is the problem of decisions. Zadeh defined the decision procedure was follows: "The search for the optimum under the worst conditions". It may be noted as a critical comment that whereas the conditions are fuzzy, the result of the decision is sharp and clearly defined. The principle formulated above says noting about the nature of the optimum.

If we wish to define a group of alternatives, then this is done via the designation of an arbitrary standard independent of the set function and the operator.

It is well known that, on the basis of the above critical observations, the theory of fuzzy sets has undergone certain changes. Various trends have developed, for example *R*-fuzzy algebra based on the operators, but these solutions have disrupted the unity of the theory. Accordingly, the literature on fuzzy sets has by now become completely diversified.

After these comments I should like to turn to the real subject-matter of the lecture.

The evaluation theory presented in the following evolved as a result of attempts to solve the problems that had arisen in the theory of fuzzy sets. I shall attempt to give satisfactory answers to the critical comments that have been mentioned, and also to certain problems of multicriteria decisions and many-valued logic.

2. Basic concepts of the evaluation theory

To return to the fuzzy concepts mentioned in the critical comments, it may be observed that each of them consists of an object and the evaluation of this according to some property, even the pragmatic interpretation of the "young factory". We shall subsequently interpret the membership function in this way as well. The evaluation of the objects in accordance with certain of their properties always plays an essential role from the point of view of the making of some decision.

In the following the term "decision" will be taken to mean that we wish to divide the objects into two groups on the basis of the elementary evaluations of their properties; either they satisfy an expectation level, or they lie below this level. However, a decision can only be made if the evaluation of the objects on the basis of their properties can be aggregated in some way, that is an evaluation operator must be constructed. The solution of problems of this type is dealt with by the theory of multicriteria decisions, but form-identification tasks in medical diagnostics are also of the same nature.

The evaluation operator can be constructed in a very large number of ways, and we shall therefore examine only operators satisfying certain conditions. As regards the evaluation operator, it will be assumed that $\sigma:[0,1]\times[0,1]\to[0,1]$, that is we shall first consider an evaluation operator that has two arguments.

3. Axiom system relating to the evaluation operator

1. The evaluation operator is continuous. It is noted here that the continuity is demanded with the exception of the points (0, 1) and (1, 0). The evaluation of the totally polar values can be interpreted in various ways.

2. The operator satisfies the requirement relating to the Pareto optimum.

$$\text{If } x_2 < x_2' \text{ then } \sigma(x_1, x_2) < \sigma(x_1, x_2'), \tag{1}$$

if $x_1, x_2 \in (0, 1)$, and monotonously in the interval [0, 1].

3. The evaluation is independent of the grouping of the properties, that is the operator is associative. This may be regarded as a criterion of objectivity:

$$\sigma(x_1, \sigma(x_2, x_3)) = \sigma(\sigma(x_1, x_2), x_3). \tag{2}$$

4. It holds for the operator that

$$\sigma(0, 0) = 0,$$
$$\sigma(1, 1) = 1. \tag{3}$$

Depending on the choice of axiom 5, we shall speak of different evaluation operators.

5(a) $\sigma(0, 1) = 0,$

5(b) $\sigma(0, 1) = 1.$ (4)

Operators satisfying conditions 5(a) or 5(b) are termed logical operators. If 5(a) holds, we speak of a conjunctive operator, denoted by $c(x,y)$, and if 5(b) holds, of a disjunctive operator denoted by $d(x,y)$. If a property assumes the value 0 in the case of a conjunctive operator, then the value 0 is assigned to the object, that is an *exclusive or screening* character is manifested. If the value 1 is assigned to one of the properties in the case of a disjunctive operator, this definitely means the best evaluation, that is an *accentuating* character is manifested.

(5c) The operator satisfies the condition of correct cluster formation. Such operators are termed aggregative operators, denoted by $a(x,y)$. For a more exact explanation of axiom 5(c), a few concepts must be introduced. We shall first speak of the negation operator. This operator satisfies the following axioms:

1. it is continuous,
2. it is strictly monotonously decreasing,
3. $n(0) = 1, \quad n(1) = 0,$ (5)
4. $n(n(x)) = x.$ (6)

For every such negation operator there exists one and only one value ν for which

$$n(\nu) = \nu. \tag{7}$$

Since this value and its negated form are the same, it may be termed a neutral value. Further, since the negation of values smaller than the neutral value gives values larger than the neutral value, and vice versa, the neutral value naturally divides the evaluation interval into two parts. The values larger than ν may be interpreted as the positive or acceptable evaluation range, and those smaller than ν as the negative evaluation range; ν is thus a threshold value, and can be interpreted as an expectation level.

After this digression, let us turn to the formulation of the condition for correct cluster formation.

If evaluation is now performed with the aid of some operator, then the acceptable and unacceptable objects may be differentiated on the basis of the neutral value ν and the aggregation values. If the aggregation value relating to an object is larger than the neutral value, then the object is placed in class C_1; if not, it is placed in class C_2. To a first approximation, axiom 5(c) may now be formulated in the following way: if the object belongs in class C_1, then when all the evaluations based on the various properties with regard to some aggregative operator are substituted by their negated forms, the aggregate of these new values belong to class C_2.

Now let us consider a set of objects $(O_1, O_2, \ldots, O_n)$. Let us characterize every object with a number m of its properties $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$, where $x_i \in (0, 1)$ and $i = 1, \ldots, n$. Thus, if the aggregative operator is denoted as $a(x_1, \ldots, x_n)$, for a decision level δ we have

$$C_{\delta,1} = \{O_i \mid a(x_{i_1}, \ldots, x_{i_m}) > \delta\},$$
$$C_{\delta,2} = \{O_i \mid a(x_{i_1}, \ldots, x_{i_m}) < n(\delta)\}. \tag{8}$$

Let us next substitute every property by its antithetic one (in the following its negative form $n(x_i)$) and carry out division into classes at the level:

$$\bar{C}_{\delta,1} = \{O_i \mid a(n(x_{i_1}), \ldots, n(x_{i_m})) > \delta\},$$
$$\bar{C}_{\delta,2} = \{O_i \mid a(n(x_{i_1}), \ldots, n(x_{i_m})) < n(\delta)\}. \tag{9}$$

The condition of correct cluster formation is thus

$$C_{\delta,1} = \overline{C}_{\delta,2}, \qquad C_{\delta,2} = \overline{C}_{\delta,1}.$$

Silvert [6] refers to this type of aggregation as symmetrical summation.

Theorem 1. *It is a necessary and sufficient condition of the aggregative operator satisfying correct cluster formation that*

$$a(x, y) = n(a(n(x), n(y))) \tag{10}$$

should hold.

Proof. *Necessity:* Let us assume that for some x, y,

$$a(x, y) < \delta.$$

Since $n(x)$ decreases strictly monotonously,

$$n(a(x, y)) > n(\delta).$$

Utilization of the condition gives

$$a(n(x), n(y)) > n(\delta)$$

which yields the definition of the other class.

Sufficiency: Let us assume that

$$n(a(x, y)) > a(n(x), n(y)).$$

Then there exists a δ for which

$$n(a(x, y)) > \delta > a(n(x), n(y)).$$

Making use of the fact that $n(n(x)) = x$ together with the left-hand side of the inequality and substituting into, $n(x)$, we have

$$a(x, y) < n(\delta).$$

It further holds that

$$\delta > a(n(x), n(y)).$$

Let $n(\delta) = \delta'$. From the above two inequalities we then have

$$a(x, y) < \delta',$$
$$a(n(x), n(y)) < n(\delta')$$

according to which

$$C_{\delta,1} \neq \overline{C}_{\delta,2}.$$

If $\delta = \nu$, then together with classes $C_{\delta,1}$ and $C_{\delta,2}$ we should also have

$$C_\nu = \{O_i \mid a(x_{i_1}, \ldots, x_{i_m}) = \nu\}. \tag{11}$$

For this class analysis it holds that

$$C_\nu = \overline{C}_\nu = \{O_i \mid a(n(x_{i_1}), \ldots, n(x_{i_m})) = \nu\}$$

since

$$\nu = n(\nu) = n(a(x_{i_1}, \ldots, x_{i_m})) = a(n(x_{i_1}), \ldots, n(x_{i_m})).$$

4. Operators satisfying the axiom system

The first result is that, all of the operators satisfying the axiom system can be written in the form

$$\sigma(x_1, x_2) = f(f^{-1}(x_1) + f^{-1}(x_2)) \tag{12}$$

where $f(x)$ is continuous and strictly increasing. The functions constituting the conjunctive, disjunctive and aggregative operators have the forms as shown in Fig. 1.

Thus, the following relations hold for the functions:

(a) conjunctive case

$$f_c(0) = 1,$$
$$\lim_{x \to -\infty} f_c(x) = 0;$$

(b) disjunctive case

$$f_d(0) = 0,$$
$$\lim_{x \to +\infty} f_d(x) = 1; \tag{13}$$

(c) aggregative case

$$\lim_{x \to -\infty} f_a(x) = 0, \qquad \lim_{x \to +\infty} f_a(x) = 1,$$
$$f(0) = \nu. \tag{14}$$

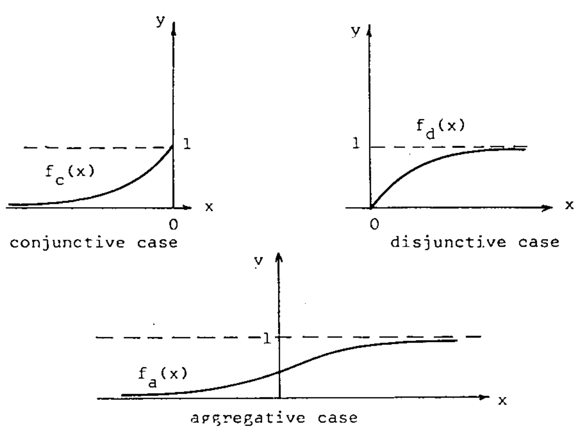

Fig. 1. The generator functions of the operators.

Logical operators have been dealt with in [2].

It should be noted that the operator is rational if, and only if, it is of the form

$$f(x)=\frac{a\,e^x+b}{c\,e^x+d} \quad\text{or}\quad f(x)=\frac{ax+b}{cx+d}. \tag{15}$$

The operators of the evaluation theory consist of these three, the conjunctive, disjunctive and aggregative operators, together with the negation operator. First let us examine the logical operators.

5. Properties of the logical operators

Let us briefly summarize the properties of these operators. The properties demanded by the axioms are clearly met by the operators, that is they are continuous, strictly monotone and associative. The correspondence principle holds for the logical operators (that is compatibility with dual logic); moreover

$$c(0,x)=0, \qquad d(1,x)=1. \tag{16}$$

Thus, the injectivity of the conjunctive operator holds in the left side open interval (0, 1], and that of the disjunctive operator in the right side open interval [0, 1).

It can further be seen that

$$c(1,x)=x, \qquad d(0,x)=x. \tag{17}$$

It follows that the operators are also commutative.

The question arises as to how the operators satisfying the axiom system compare to the min and max operators proposed by Zadeh. Firstly, for all logical operators

$$\begin{aligned} 0 &\leqslant c(x_1,x_2)<\min(x_1,x_2),\\ 1 &\geqslant d(x_1,x_2)>\max(x_1,x_2), \end{aligned} \qquad x_1,x_2\in(0,1). \tag{18}$$

Clearly, the minimum and maximum operators can not be derived from the axiom system, since they are not strictly monotone.

Mention may be made of another theorem by means of which further light may be shed on the relation to the Zadeh operators. Let $f(x)$ and $g(x)$ be functions constituting conjunctive and disjunctive operators. The functions

$$f_\lambda(x)=f(x^{1/\lambda}), \qquad g_\lambda(x)=g(x^{1/\lambda}) \tag{19}$$

also satisfy the properties of the functions constituting the conjunctive and disjunctive operators. $c_\lambda(x_1,x_2)$ and $d_\lambda(x_1,x_2)$ formed with the aid of $f_\lambda(x)$ and $g_\lambda(x)$ possess the following properties:

$$\begin{aligned} &\lim_{\lambda\to\infty} c_\lambda(x_1,x_2)=\min(x_1,x_2),\\ &\lim_{\lambda\to\infty} d_\lambda(x_1,x_2)=\max(x_1,x_2). \end{aligned} \tag{20}$$

That is, the limit operators of the operator series give the min and max operators.

6. Properties of the aggregative operators

The properties of the function generating the aggregative operator will be considered in the interval (0, 1).

On the basis of the theorem of Aczél, we know that $a(x,y)=a(y,x)$ (commutativity) holds.

Theorem 2. *It holds that*

$$a(x,n(x))=\nu. \tag{21}$$

Proof. Since

$$n(a(x,y))=a(n(x),n(y))$$

and letting $y=n(x)$

$$n(a(x,n(x)))=a(n(x),n(n(x)))=$$
$$a(n(x),x)=a(x,n(x)).$$

On the basis of the property of the negation operator, there is only one ν value for which $n(\nu)=\nu$. Thus

$$a(x,n(x))=\nu.$$

Theorem 3. *For the function generating the aggregative operator, it holds that*

$$f(0)=\nu, \qquad f^{-1}(\nu)=0, \tag{22}$$

$$\lim_{x\to 1} f^{-1}(x)=\infty \quad \textit{and} \quad \lim_{x\to 0} f^{-1}(x)=-\infty;$$

or

$$\lim_{x\to 1} f^{-1}(x)=-\infty \quad \textit{and} \quad \lim_{x\to 0} f^{-1}(x)=\infty. \tag{23}$$

Proof. Since

$$a(x,n(x))=\nu$$

and

$$a(x,y)=f(f^{-1}(x)+f^{-1}(y))$$

we have

$$f(f^{-1}(x)+f^{-1}(n(x)))=\nu.$$

Utilizing the strictly monotonous and continuous nature of $f(x)$:

$$f^{-1}(x)+f^{-1}(n(x))=f^{-1}(x).$$

Let $x=\nu$, so that

$$f^{-1}(n(\nu))=f^{-1}(\nu)=0.$$

Let us assume that

$$\lim_{x\to 1} f^{-1}(x)=c\neq\pm\infty, \quad \text{i.e.} f(c)=1.$$

Since $0<\nu<1$ and $f^{-1}(\nu)=0$, we have $c\neq 0$. Further,

$$f(c)=1,$$

$$a(1,1)=\lim_{\substack{x\to 1\\ y\to 1}} f(f^{-1}(x)+f^{-1}(y))=f(2c).$$

Because of the strictly monotone property, this is only possible if

$$c=2c$$

which is contradictory, for $c\neq 0$, $c\neq\pm\infty$. The other property can be proved similarly.

Since the generator function of the operator may be either monotonously increasing or decreasing, in the following we shall consider only strictly monotonously increasing generator functions $f(x)$.

Thus, the function $f(x)$ generating the aggregative operator possesses the following properties (see also Fig. 2):

1. $f:(-\infty,\infty)\to(0,1)$,
2. it is continuous,
3. it is strictly monotonously increasing,
4. $f(0)=\nu$,
5. $\lim_{x\to-\infty} f(x)=0, \qquad \lim_{x\to\infty} f(x)=1.$ (24)

Theorem 4. *For the aggregative operator whose generator function satisfies the above properties, it holds that*

1. *it is associative,*
2. *it is continuous, except for points* (0, 1) *and* (1, 0),
3. *it is strictly monotonously in* $(0,1)\times(0,1)$,
4. *it is commutative.*

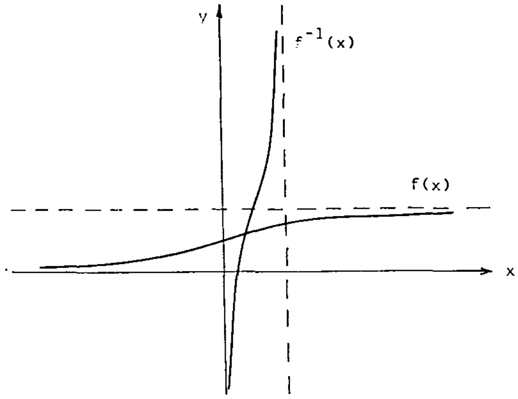

Fig. 2. The aggregative operator generating functions $f(x)$ and $f^{-1}(x)$.

Proof. See under logical operators [2].

Theorem 5. *It holds for the aggregative operator that*

1. $a(x,n(x))=\nu, \quad x\neq 0, x\neq 1,$ (25)
2. $a(x,\nu)=x,$ (26)
3. $a(x,1)=1 \quad$ *if* $x\neq 0$,
4. $a(x,0)=0 \quad$ *if* $x\neq 1$. (27)

Proof. 1. See Theorem 2.

2. $a(x,\nu)=f(f^{-1}(x)+f^{-1}(\nu))=f(f^{-1}(x))=x.$

3. $x\neq 0$, therefore $f^{-1}(x)>-\infty$, and thus $a(x,1)=\lim_{y\to 1} f(f^{-1}(x)+f^{-1}(y))=\lim_{z\to\infty} f(z)=1.$

4. $x\neq 1$, therefore $f^{-1}(x)<\infty$, and thus $a(x,0)=\lim_{y\to 0} f(f^{-1}(x)+f^{-1}(y))=\lim_{z\to-\infty} f(z)=0.$

Theorem 6. For the aggregative operator, it holds that

1. *if* $x,y\leqslant\nu$, *then* $\min(x,y)\geqslant a(x,y)$,
2. *if* $x,y\geqslant\nu$, *then* $a(x,y)\geqslant\max(x,y)$,
3. *if* $x=\nu\leqslant y$, *then* $\min(x,y)\leqslant a(x,y)\leqslant\max(x,y)$. (28)

Proof. Let us assume that $x\leqslant y$ in all three cases, and let us utilize the strictly monotonously increas-

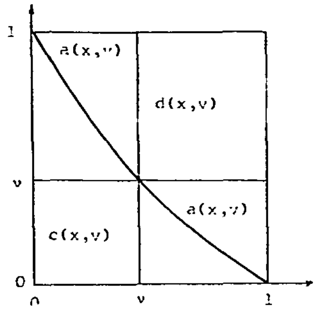

Fig. 3. Regions of the aggregative operator.

ing nature of the operator:

1. $y \leqslant \nu$, therefore $a(x, y) \leqslant a(x, \nu) = x = \min(x, y)$,
2. $x \geqslant \nu$, therefore $a(x, y) \geqslant a(\nu, y) = y = \max(x, y)$,
3. $x \leqslant \nu \leqslant y$, therefore $\min(x, y) = x = a(x, \nu) = a(x, y) \leqslant a(\nu, y) = \max(x, y)$.

The above considerations are illustrated in Fig. 3.

7. Formation of the aggregative operator

Theorem 7. *For the aggregation of a number of arguments, it holds that*

$$a(x_1,\ldots,x_n) = f\left(\sum_{i=1}^{n} f^{-1}(x_i)\right). \tag{29}$$

Proof. The formula follows from the basic form of the aggregative operator.

Theorem 8. *If $f(x)$ is the generator function of the aggregative operator, then the function displaced by d, $f(x+d) = f_d(x)$, also possesses the properties of the generator function.*

Proof. The continuity, the strict monotone property and the properties $\lim_{x\to 0} f^{-1}(x) = -\infty$ and $\lim_{x\to 1} f^{-1}(x) = +\infty$ are realized trivially.

The neutral value ν naturally varies, which allows the formation of aggregative operators with different neutral values from one generator function.

Theorem 9. *For the displacement relating to the aggregative operator with neutral value ν, it holds that*

$$d = f^{-1}(\nu). \tag{30}$$

Proof. Since $f_d(x) = f(x+d)$ and $f_d(0) = \nu$, we have $f(d) = \nu$, from which $d = f^{-1}(\nu)$.

In the following, let us denote the aggregative operator possessing the neutral value ν by $f_\nu(x)$. Hence, on the basis of the above propositions

$$f_\nu(x) = f(x + f^{-1}(\nu)). \tag{31}$$

Let us generalize the multiargument aggregative relationship.

Theorem 10. *For the multiargument aggregative operator with neutral value ν, it holds that*

$$a(x_1,\ldots,x_n) = f\left(\sum_{i=1}^{n} f^{-1}(x_i) - (n-1) f^{-1}(\nu)\right). \tag{32}$$

Proof. This follows from the previous proposition and the fact that

$$f_\nu^{-1}(x) = f^{-1}(x) - f^{-1}(\nu) \tag{33}$$

by substitution into the formula for the multiargument aggregation.

Let us denote by A the value obtained by the aggregation of an object. In the knowledge of this, the neutral value can be determined in the following way:

$$\nu = f\left(\frac{\sum_{i=1}^{n} f^{-1}(x_i) - f^{-1}(A)}{n-1}\right). \tag{34}$$

For the aggregative operator the neutral value plays an important role. The following relations hold:

$$a(x, n(x)) = \nu,$$
$$a(\nu, x) = x. \tag{35}$$

The above two relations effectively support the neutral character of ν. The following inequalities also hold for the aggregative operator:

(a) If $x_1, x_2 \leqslant \nu$, then $a(x_1, x_2) \leqslant \min(x_1, x_2)$;

(b) If $x_1, x_2 \geq \nu$, then $a(x_1, x_2) \geq \max(x_1, x_2)$;

(c) If $x_1 \leq \nu \leq x_2$ then $\min(x_1, x_2) \leq a(x_1, x_2) \leq \max(x_1, x_2)$.

This accentuates the character of ν as a threshold value or expectation level. According to (a), the negative values weaken each other, (b) the positive values strengthen each other, and (c) an aggregation of positive and negative values results in a compromise.

If the inequalities (a) and (b) are compared with the inequalities relating to the logical operators, the hypothesis arises that below the neutral value the aggregation is a conjunction, and above it a disjunction; this is also indicated by the facts that

$$a(x, 0) = 0, \quad x \neq 1,$$

$$a(x, 1) = 1, \quad x \neq 0.$$

Our hypothesis may be formulated in the statement that the conjunctive and disjunctive operators are simply variations of the neutral value, or transformations of the elementary evaluations, above and below the neutral value.

What neutral value can be assigned to the conjunctive and disjunctive operators? Since it holds for the aggregative operator that

$$f_a(0) = \nu \tag{36}$$

the interpretation of the neutral value can also be extended in this way. In the conjunctive case this value is 1, while in the disjunctive case it is 0. Since the neutral value is also an expectation level, it may be said that in the case of a conjunctive operator the expectation level assigned to the decision is maximum, in a disjunctive case there is no expectation at all, while the aggregative operator is situated between the two logical operators.

8. Illustration of the operators

Let us illustrate the various decision procedures with everyday examples. The buying of a *HiFi* system can be given as an example of a conjunctive decision procedure, for the optimum value of every parameter can then be defined, and in comparison to this the parameters of all real appliances can only be poorer. In fact, therefore, the goodness value of the parameters of an appliance actually indicate the badness.

An example of a disjunctive decision is the case where we may select something as a present from a set of given objects. In this case no expectation level can be established. (Never look a gift-horse in the mouth. Beggars can't be choosers.) Here we come to a decision in accordance with individual aspects, and good and better really do exist.

The application of the aggregative operator perhaps occurs most often in practice. If the Head of a Department must evaluate his staff, he cannot proceed as in the making of a conjunctive decision, by assuming that each of them possesses only "negative" properties compared to a theoretically established ideal expectation level. Nor can be apply a disjunctive model, defining no expectation level and assuming that every individual displays only good and better properties. Instead, he must evaluate this staff in comparison to a predetermined expectation level.

To remain for a little with the philosophic aspect, two mutually opposing tendencies may generally be observed in decision making. The basis of an individual decision improves as the expectation level approaches the ideal, that is as ν increases. On the other hand, since the rise of the expectation level is accompanied by an appreciable number of problems (acquisition of information, determination of ideal values, etc.), everyone making decisions attempts to use as little energy as possible for the later decisions in the course of a series of decisions. This requires that the restriction of the conditions should be reduced, and that ν should be decreased.

The three operators of the evaluation theory can be treated uniformly. Their differences lie in the differences of the expectation levels. It should be noted that to date the theories have only examined the conjunctive model.

9. The derivation of homogeneous logical systems

To return to the operators, in the case of logical operators an essential role has always been played by the DeMorgan identity. We have studied [2] the question of what relationship exists between the general solution of the axiom system and the types of functions generating logical operators which satisfy the DeMorgan identity. It may be stated that if the DeMorgan identity holds, the following relationship exists between the functions constituting the operators:

$$n(x)=f_c(f_d^{-1}(ax)) \tag{37}$$

where $f_c(x)$ and $f_d(x)$ are the functions that generate the conjunctive and disjunctive operators, respectively. The above condition is necessary and sufficient. The relation allows any two logical operators to be used to construct the third, so that the DeMorgan identity should hold.

The existence of the DeMorgan identity, and the conjunctive and disjunctive operators, determine a negation [2].

Let us examine the relationship between the aggregative operator and the negation operator.

Theorem 11. *For the negation operator relating to the aggregative operator, it holds that*

$$n(x)=f(-f^{-1}(x)). \tag{38}$$

Proof.

$$a(x, n(x))=f(f^{-1}(x)+f^{-1}(n(x)))=\nu.$$

Therefore

$$f^{-1}(x)+f^{-1}(n(x))=f^{-1}(\nu)=0.$$

From which

$$f^{-1}(n(x))=-f^{-1}(x).$$

By expressing $n(x)$, we obtain the above proposition.

Let us denote the negation relative to the neutral value ν by $n_\nu(x)$.

Theorem 12.

$$n_\nu(x)=f(2f^{-1}(\nu)-f^{-1}(x)). \tag{39}$$

Proof. It is known that

$$f_\nu(x)=f(x+f^{-1}(\nu)) \quad \text{and}$$

$$f_\nu^{-1}(x)=f^{-1}(x)-f^{-1}(\nu).$$

If this is substituted into $n(x)=f(-f^{-1}(x))$, we obtain the proposition.

Theorem 13. *For the negation defined by* $n(x)=f(-f^{-1}(x))$, *it holds that*

1. *it is continuous,*
2. *it is strictly monotonously decreasing,*
3. $n(n(x))=x$.

Proof. The continuous nature of $n(x)$ follows from the continuous natures of $f(x)$ and $f^{-1}(x)$.

Let $x<y$. It then follows from the strictly monotonously increasing nature of $f^{-1}(x)$ that

$$f^{-1}(x)<f^{-1}(y).$$

Thus,

$$-f^{-1}(x)>-f^{-1}(y).$$

Since $f(x)$ is strictly monotonously: increasing,

$$n(x)=f(-f^{-1}(x))>f(-f^{-1}(y))=n(y),$$

i.e. $n(x)$ is strictly monotonously decreasing. We now show that $n(1)=0$:

$$n(1)=\lim_{x\to 1} f(-f^{-1}(x))=\lim_{z\to\infty} f(z)=0.$$

It can similarly be seen that $n(0)=1$. Finally, we see that $n(n(x))=x$:

$$\begin{aligned} n(n(x)) &= f(-f^{-1}(n(x))) \\ &= f(-f^{-1}(f(-f^{-1}(x)))) \\ &= f(-(-f^{-1}(x)))=x \end{aligned}$$

From the continuous strictly monotonous nature of $n(x)$ and from the fact that it assumes values of 0 and 1, it follows that it maps from the interval [0, 1] to the interval [0, 1].

10. The rational class of aggregative operators and some formations of them

Theorem 14. *The class of rational aggregative operators*

$$a(x_1, x_2,\dots,x_n)=\left[(1-\nu)^{n-1}\prod_{i=1}^{n} x_i\right] \left[(1-\nu)^{n-1}\prod_{i=1}^{n} x_i+\nu^{n-1}\prod_{i=1}^{n}(1-x_i)\right]^{-1} \tag{40}$$

$x_i=0$ *and* $x_i=1$ *can not hold simultaneously.*

Proof. On the basis of the theorems of Alt and Kuwagaki, the generator functions of the rational solutions of the associative function equation are of the form

$$f(x)=\frac{a+bx}{c+dx}, \qquad f(x)=\frac{a+b\,e^x}{c+d\,e^x}. \tag{41}$$

Since it holds for the generator function $f(x)$ that $\lim_{x\to-\infty}f(x)=0$ and $\lim_{x\to\infty}f(x)=1$, for the parameters of the linear fractional function the equations

$$\frac{b}{d}=1 \quad \text{and} \quad \frac{b}{d}=0$$

must hold.

This is not possible. For the second generator function, we have

$$\frac{b}{d}=1 \quad \text{and} \quad \frac{a}{c}=0.$$

Introducing $k=b/c$, we obtain the generator function

$$f(x)=\frac{k\,e^x}{1+k\,e^x}.$$

The proposition is confirmed by using the general form of the multiargument aggregative operator with neutral value ν.

Let $\nu=0.5$ and $n=2$. We then obtain the Luce selection formula. From knowledge of the aggregated value A, the value can be determined. For the sake of simplicity in writing, let $c=\nu/(1-\nu)$. Then

$$c=\left(\frac{\prod x_i}{\prod(1-x_i)}\cdot\frac{1-A}{A}\right)^{1/(n-1)} \tag{42}$$

where $A\neq 0$ and $x_i\neq 1$.

Theorem 15. *For the negation relating to the rational aggregative operator with neutral value ν, it holds that*

$$n(x)=\frac{\nu^2(1-x)}{(1-\nu)^2x+\nu^2(1-x)}. \tag{43}$$

Proof. The assertion may be obtained by substituting the generator function of the rational aggregative operator into the negation operator expression.

Note. Let us perform the parameter transformation

$\gamma=((1-\nu)/\nu)^2-1$ on $n(x)$. Then

$$n(x)=\frac{1-x}{1+\gamma x}. \tag{44}$$

This form agrees with the negation operator introduced by Hamacher, Sugeno.

Further aggregative operators may be obtained by different selections of $f(x)$. Without any proof, we give an algebraic and an irrational solution.

Let

$$f(x)=\tfrac{1}{2}\left(\left(1-\frac{1}{x}\right)+\sqrt{\left(1-\frac{1}{x}\right)^2+\frac{4\nu}{x}}\right),$$

$$f^{-1}(x)=\frac{x-\nu}{x(1-x)}. \tag{45}$$

If $\nu=0.5$, then

$$a_{0.5}(x,y)=\tfrac{1}{2}\left(1-K+\sqrt{1+K^2}\right),$$

where

$$K=\frac{2xy(1-x)(1-y)}{y(1-y)(2x-1)+(1-x)(2y-1)}. \tag{46}$$

For the negation relating to $f(x)$

$$n(x)=\tfrac{1}{2}\left(1+L+\sqrt{(1+L)^2-4\nu L}\right),$$

$$\text{where } L=\frac{x(1-x)}{x-\nu}. \tag{47}$$

Let

$$f(x)=\frac{1}{\pi}\arctan(x)+\tfrac{1}{2},$$

$$f^{-1}(x)=\tan(\pi(x-\tfrac{1}{2})). \tag{48}$$

Thus, for the aggregative operator, we have

$$a(x_1,x_2,\ldots,x_n)=\frac{1}{\pi}\left(\arctan\left(\sum_{i=1}^{n}\tan\pi(x_i-\tfrac{1}{2})-(n-1)\tan\pi(\nu-\tfrac{1}{2})\right)\right)+\tfrac{1}{2} \tag{49}$$

while for the related negation operator

$$n(x)=\frac{1}{\pi}\arctan(2\tan\pi(\nu-\tfrac{1}{2})-\tan\pi(x-\tfrac{1}{2})). \tag{50}$$

Since the aggregative operator is derived with the aid of the negation operator, it automatically induces a negation:

$$n(x)=f_a(-f_a^{-1}(x)).$$

We have also seen that a connection can be created between an aggregative operator and logical operators, and hence the logical operators can be derived from the aggregative operator. Thus, the aggregative operator determines a logical system. Overall, therefore, a logical system can be derived from a function chosen to generate the aggregative operator. With various choices of the function, known logical systems may be obtained, for example those of Hamacher, Yager, Silvert, Zadeh, etc.

I should like to mention merely the main areas in which I have attained new results with the aid of the theory.

11. The aggregate implications

From the aggregative operator a new implication can be derived, which is equivalent to the Zadeh definition of set implication. More exactly, the value assigned to the implication operation is larger than the threshold value ν if, and only if, the Zadeh implication holds for the elementary evaluations. The implication permits a simple generalization of the syllogisms; the most varied attempts that have previously been made at this in the theory of fuzzy sets have been by means of different relations.

12. The decision measures

With the aid of the functions constituting the operators, I have given the extent of separation of the two classes obtained via the aggregative operator; this is a generalization of the earlier fuzziness measures. I have also extended these measures to the logical operators. In this way we have not only defined the decision procedure, but also given a measure of the sharpness of the decision.

13. The scale invariance

If an evaluation theory is to be employed in the field of multicriteria decisions, then since the elementary evaluations may belong to different scales, it is of great importance to know to what scale transformations the value of the aggregation is invariant. The evaluation is said to be invariant if the values of the aggregates before and after transformation are in the same relation to one another. I have likewise determined a relation using the function constituting the operator. I have given the interdependence of the general scale invariance.

14. The problem of weighting

In the theory I have separated the evaluation and the importance of the properties. The whole process requires the consideration of both the evaluation and its importance. We have modified the general construction of the operators, in two ways:

(1) with the said of an importance relation, and

(2) on the basis of a repetition principle.

This latter means that if some property is more important than the others, it may be accentuated by repetition of the argument, while those that are insignificant are not mentioned. In the former case the importance wights may also be determined implicitly, by solution of a linear equation system. If the sum of the weights is 1, then the evaluation is independent of the neutral value. To mention a few examples:

If $f(x)=e^{-x}$, then the aggregation is $\prod x_i$, the weighting is $\prod x_i^{w_i}$, and the scale invariance is $x_i^*=c_i x_i$.

If $f(x)=x$, then the aggregation is $\sum x_i$, the weighting is $\sum w_i x_i$, and the scale invariance is $x_i^*=x_i+c_i$.

If $f(x)=1-e^{-x}$, then the aggregation is $1-\prod(1-x_i)$, the weighting is $1-\prod(1-x_i)^{w_i}$, and the scale invariance is $x_i^*=(1-c_i)+x_i c_i$.

The first two examples are in accordance with utility theory.

To return to the initial critical comments

(1) The membership function is an evaluation which must be given with consideration to the neutral value. The invariance and the aggregation are closely interdependent.

(2) We have not fixed the operators; they satisfy a weak axiom system.

(3) The results of the decision procedure is not a sharp set. The decision is interrelated with the axiom system. Not only is the decision defined,

but the measure of sharpness of the decision is given too.

(4) The operators compromise a homogeneous system, including set implication.

The evaluation theory can be employed widely in practice. Finally, I should like to mention two interesting features:

(1) The aggregative operator is not a logical operator, that is, it can not be obtained as an expression of logical operators, but logical operator systems can be derived from it.

(2) The sharpness of the operators can be defined too, and from this aspect the operators of the theory of fuzzy sets are the fuzziest operators.

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